

## A SIMPLE AND APPARENTLY SAFE SOLUTION TO THE GENERALISED STEFAN PROBLEM

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**Abstract**—Solidification or melting of a slab-shaped body can occur under various conditions on both boundaries of the solid. Only the simplest situation has been subjected to a rigid analysis whereas the more general problem has been solved by approximate theories and by numerical simulation. However, most of the results are inconvenient for engineering purposes, such as optimization of phase-change equipment, and the theories predict rates of phase change which are too small in some cases and too large in others.

The aim of this article is to provide a result which is of a simple analytic structure and which predicts too long (safe) freezing times for all parameter combinations of practical interest. Further, it is shown that this result can be employed to generate working equations of high accuracy.

### NOMENCLATURE

$Bi$ ,	$= h_c \delta_0 / k$ , Biot number;
$c$ ,	specific heat capacity;
$h$ ,	heat-transfer coefficient;
$k$ ,	thermal conductivity;
$Ph$ ,	$= \lambda' / [c(T_p - T_c)]$ ,
	phase-change parameter;
$t$ ,	time;
$T$ ,	temperature;
$x$ ,	co-ordinate in the direction of the moving interface.

### Greek letters

$\alpha$ ,	thermal diffusivity;
$\delta$ ,	thickness of solidified layer;
$\delta^*$ ,	$= \delta / \delta_0$ , dimensionless thickness of solidified layer;
$\Delta$ ,	$= h_m(T_m - T_p) / h_c(T_p - T_c)$ , heat-flux parameter;
$\Delta^*$ ,	$= \Delta Bi$ , modified heat-flux parameter;
$\lambda$ ,	latent heat of fusion;
$\rho$ ,	density;
$\tau$ ,	$= t\alpha / \delta^2$ , dimensionless time.

### Subscripts

$c$ ,	relating to cooling fluid;
$m$ ,	relating to melt;
$p$ ,	at phase-change conditions;
$w$ ,	relating to cooling wall;
$0$ ,	at reference time $t_0$ .

### Superscripts

- (1), "upper" limit of solidification time;
- (2), lower limit of solidification time;
- (3), weighted mean of solidification time.

### 1. INTRODUCTION

PROCESSES involving solid-liquid phase change are encountered in the chemical, food and metallurgical industry as well as in fields such as surgery, cryobiology and aerospace. Applications include the formation of polar ice [1], the production of chemicals as prills [2] or flakes [3], the freezing and destruction of tumorous

brain tissue [4], the continuous casting of metals [5] and many other operations. In all these cases the most important problem is to establish the position of the solid-liquid interface as a function of time, external conditions and physical as well as geometric properties of the system. With the solution at hand one can immediately specify main process parameters such as the height of a prilling tower, the rotational speed of a flaker, the temperature of a cryoprobe or length of a cooling mould for casting. If, in addition, the solution to the phase-change problem is of a simple analytic structure one can also devise optimum operating conditions for a given system or an optimum design for a given operation. This is of particular interest in some of the processes mentioned above.

Difficulties in the mathematical treatment of the problem arise primarily from the boundary conditions at the moving interface and exact analytic solutions exist only for some limiting cases. If the temperature of the cooling wall is constant and the liquid is at fusion temperature one obtains the Stefan-Neumann solution [6]; if the change in internal energy of the solid layer is negligibly small one obtains the quasi-stationary solution [7-9]. The more general case of finite heat transfer at the boundaries of the solid as well as sizeable change in its internal energy during the process has been treated in two different ways:

Analyses are based on the assumption of a second- or higher-order polynomial for the temperature distribution in the solid; approximate solutions are then obtained by applying variational methods [10], integral techniques [11, 12] or by satisfying the governing equation at the moving interface only [13-15]. However, it is a shortcoming of all these methods that the absolute error of the result is not known, does not necessarily decrease with higher order of the polynomial (see the discussion in [16]) and may vary in sign [14, 17]. Still, over a significant range of parameter values good accuracy can be achieved if one is prepared to work with rather complicated and lengthy expressions, some of which can only be evaluated on the computer.

The second class of solutions has been developed by numerical simulation [17–21] and is presented in diagrams. The results are usually very accurate and are therefore useful for testing the validity of approximate methods. However, numerical results are not suitable for the purposes mentioned above. They lack the required analytic structure and in some places the diagrams can only be read to 10% accuracy.

In summary, we therefore believe that there is a place for a simple analysis which provides safe results, is sufficiently accurate over an important range of parameters values and can be used to generate equally simple results of improved accuracy. As the approach developed below breaks down for cylindrical and spherical systems we shall only consider the slab geometry; this foregoing should also provide a simple and clear notation. A modified analysis for other geometries will be presented at a later stage [22].

## 2. CONCEPT AND ANALYSIS

In the following we shall deal exclusively with the freezing of a liquid in front of a plane wall. At the end it will be outlined under which conditions the results also apply to the corresponding melting problem. Constant physical properties are assumed throughout and the solidification process may occur between the two constant temperatures of melt and cooling fluid.

According to Fig. 1 there is a linear temperature distribution in the solidified layer at any time. From

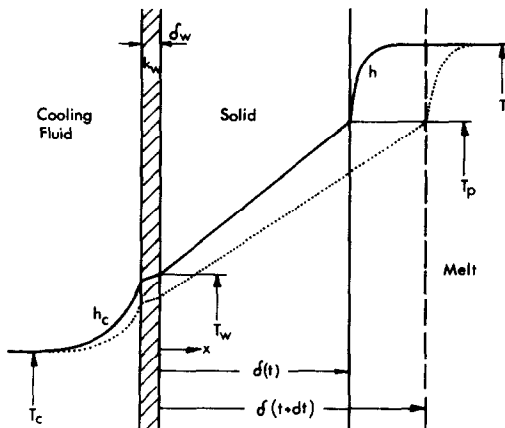


FIG. 1. Assumed temperature distribution during solidification.

the one-dimensional Laplace equation or from an energy balance one obtains immediately the quasi-stationary solution. However, it is seen that the calculated freezing time for a given layer thickness is always too small because the removal of internal energy from the solid has been neglected. Hence this solution provides a “lower bound” for the freezing time.

Alternatively one can calculate the freezing time under the condition that after each time increment internal energy is removed until the temperature profile has again relaxed to the steady-state (linear) one. This should result in an “upper bound” for the

freezing time, a statement which may require some justification.

Initially the solidification front must stay behind that of the true process because latent heat cannot be removed at the same rate. On the other hand, the final position of the phase front is the same in both the fictitious and the true processes so that at some stage the former must proceed faster than the latter. The critical question is whether, at any point in time, the calculated phase front can actually overtake the true one in which case our analysis might predict too short a freezing time. The conclusion here is that this is not possible although a formal mathematical proof cannot be presented for the general problem. However, the following development will show that the above conjecture holds for the classical Stefan problem and is likely to also hold for the other types of boundary conditions.

With the symbols defined in the nomenclature and illustrated in Fig. 1 the general problem is formulated as follows:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \quad (1)$$

$$k \frac{\partial T}{\partial x} \Big|_{x=0} = h'_c (T_w - T_c) \quad (2)$$

and

$$k \frac{\partial T}{\partial x} \Big|_{x=\delta} = \rho \lambda' \frac{d\delta}{dt} + h_m (T_m - T_p). \quad (3)$$

Here we have neglected the change in internal energy of the cooling wall and combined the film resistance  $1/h_c$  and wall resistance  $\delta_w/k_w$  in the overall coefficient

$$h'_c = \left( \frac{1}{h_c} + \frac{\delta_w}{k_w} \right)^{-1}. \quad (4)$$

Further, any desuperheating of the melt is, with good approximation [23], incorporated in the latent heat such that

$$\lambda' = \lambda + c_m (T_m - T_p). \quad (5)$$

Integrating equation (1) over  $\delta$  yields

$$\int_0^\delta \frac{\partial T}{\partial t} dx = \alpha \left( \frac{\partial T}{\partial x} \Big|_{x=\delta} - \frac{\partial T}{\partial x} \Big|_{x=0} \right). \quad (6)$$

With Leibniz's rule

$$\frac{d}{dt} \left\{ \int_0^\delta T dx \right\} = \int_0^\delta \frac{\partial T}{\partial t} dx + T(\delta) \frac{d\delta}{dt}$$

and the boundary conditions (2) and (3) it follows that

$$\left( \frac{\lambda'}{c} + T_p \right) \frac{d\delta}{dt} = \frac{d}{dt} \int_0^\delta T dx + h'_c (T_w - T_c) / (\rho c) - h_m (T_m - T_p) / (\rho c). \quad (7)$$

Integrating over the time  $\bar{t}$  (when the solidified layer

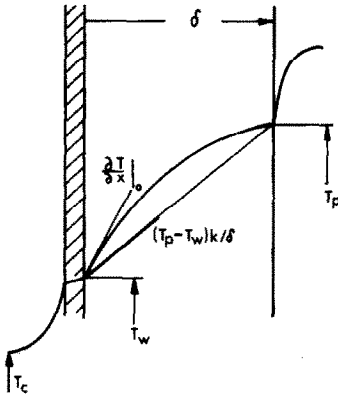


FIG. 2. Illustration of the simplifying assumptions.

thickness is just  $\delta$ ) leads to

$$\left(\frac{\lambda'}{c} + T_p\right)\delta = \int_0^\delta T \, dx + \frac{h'_c}{\rho c} \int_0^{\bar{\tau}} (T_w - T_c) \, dt - \frac{h_m \bar{\tau}}{\rho c} (T_m - T_p). \quad (8)$$

At this stage the following approximations (illustrated in Fig. 2) may be made:

$$(a) \quad \int_0^\delta T \, dx \geq \frac{T_p + T_w}{2} \delta, \quad (9)$$

$$(b) \quad \frac{h'_c}{\rho c} (T_w - T_c) \geq \frac{\alpha}{\delta} (T_p - T_w). \quad (10)$$

Thereby

$$\delta \left\{ \frac{\lambda'}{c(T_p - T_c)} + \frac{1}{2(1 + k/(h'_c \delta))} \right\} \geq \frac{h'_c}{\rho c} \int_0^{\bar{\tau}} \frac{1}{1 + h'_c \delta/k} \, dt - \frac{h_m \bar{\tau}}{\rho c} (T_m - T_p) \quad (11)$$

For convenience the following dimensionless quantities are introduced:

$$\left. \begin{aligned} \tau &= t\alpha/\delta^2; & \delta^* &= \delta/\delta_0; & Bi &= h'_c \delta_0/k; \\ Ph &= \frac{\lambda'}{c(T_p - T_c)}; & \Delta &= \frac{h_m(T_m - T_p)}{h'_c(T_p - T_c)}. \end{aligned} \right\} \quad (12)$$

The reference length  $\delta_0$  is, of course, arbitrary and one can choose other ones such as  $k_s/h'_c$  [21] or the maximum layer thickness  $L$  at infinite time [17]. However, this means changing the notation of the variables  $\tau$  and  $\delta^*$  when either the wall temperature is constant ( $h'_c \rightarrow \infty$ ) or the convective flux from the melt is zero ( $L \rightarrow \infty$ ). In the present notation the time  $\tau_0$  is that to solidify a layer of specified thickness  $\delta_0$  ( $\delta^* = 1$ ). For constant wall temperature the Biot parameter is  $Bi = \infty$  and one can simply define a parameter  $\Delta^* = \Delta Bi$  which stays finite. The limiting thickness at infinite time  $\tau_0$  is given by

$$Bi = \frac{1}{\Delta} - 1; \quad (13)$$

at this value of  $\delta_0$  the energy transferred from the melt to the solid can just be removed by the cooling fluid and no capacity is available for removing latent heat.

With the variables and parameters so defined the relation (11) becomes

$$\left\{ \frac{Ph}{Bi} + \frac{\delta^*}{2(1 + Bi\delta^*)} \right\} \delta^* \geq \int_0^{\bar{\tau}} \frac{1}{1 + Bi\delta^*} \, d\tau - \Delta \bar{\tau}. \quad (14)$$

It is important to notice that the inequality (14) still holds for the true process; it is only now that we introduce a further approximation by defining a process  $\delta^{*(1)}(\tau)$  which is governed by the equality

$$\left\{ \frac{Ph}{Bi} + \frac{\delta^{*(1)}}{2(1 + Bi\delta^{*(1)})} \right\} \delta^{*(1)} = \int_0^{\bar{\tau}} \frac{1}{1 + Bi\delta^{*(1)}} \, d\tau - \Delta \bar{\tau}. \quad (15)$$

Apparently this equation exactly expresses the physical concept proposed above. The transition from equation (14) to (15) is clearly achieved with a  $\delta^{*(1)}(\tau)$  that is "on average" not larger than the true  $\delta^*(\tau)$ ; also, it was shown before that at least in the early stage of the process  $\delta^{*(1)}(\tau)$  is smaller than  $\delta^*(\tau)$ . However, in principle one cannot exclude the possibility that temporarily  $\delta^{*(1)}(\tau)$  becomes larger than  $\delta^*(\tau)$  although such a deviation, if possible at all, would remain small because of the integral relationship.

Justification for our conjecture is obtained primarily from the comparison with numerically exact results and with the analytic result for the classical Stefan problem. In the latter case

$$\delta_0 = C_1(Ph)\sqrt{\alpha t_0} \quad (16)$$

whereas equation (14) yields

$$\delta_0^{(1)} = C_2(Ph)\sqrt{\alpha t_0}. \quad (17)$$

Since  $C_2(Ph)$  is always smaller than or equal to  $C_1(Ph)$  it follows that equation (15), adequately simplified, must provide an upper bound for the time of solidification. In all other cases an analytic relationship equivalent to equation (16) is not known; therefore a formal proof cannot be formulated on the same basis. Various other attempts to prove or disprove the bounding character of equation (15) have not been successful until now. Still, the comparison of results suggests that the concept also holds for the more general boundary conditions so that the solution of equation (15) may, with some care, be termed an upper bound.

After differentiation and rearrangement equation (15) can be integrated directly and after some lengthy but elementary calculus the solution is

$$\tau_0^{(1)} Bi^2 = (-Ph - \frac{1}{2}) \left( \frac{Bi}{\Delta} + \frac{1}{\Delta^2} \ln \left\{ 1 + \frac{Bi}{1 - 1/\Delta} \right\} \right) - \frac{1}{2} \ln \left\{ 1 + \frac{Bi}{1 - \Delta(1 + Bi)} \right\}. \quad (18)$$

The maximum time ( $\tau_0^{(1)}Bi^2$ ) to solidify a layer of thickness  $\delta_0$  thus depends on the parameters  $Ph$ ,  $Bi$  and  $\Delta$ . The lower bound for the time is found by considering very large values of  $Ph$ ; it is

$$\frac{\tau_0^{(2)}Bi^2}{Ph} = -\frac{Bi}{\Delta} - \frac{1}{\Delta^2} \ln \left\{ 1 + \frac{Bi}{1-1/\Delta} \right\}. \quad (19)$$

This is the well-known result from the quasi-stationary theory [9, 15]. Further, it is verified that by inserting equation (13) into equations (18) or (19) the time becomes infinity.

3. COMPARISON OF RESULTS

Most of the results presented in the literature are not valid over the entire range of the three parameters  $Ph$ ,  $Bi$  and  $\Delta$ . Numerically exact data have been obtained for  $\Delta = 0$  [21] and for constant wall temperature [17] whereas only one approximate analytic solution [15] is available for comparison with the general result, equations (18) and (19).

3.1. Finite heat transfer on both sides of the solidified layer

The maximum possible error involved in using equation (19) can be estimated immediately by comparing the two results (18) and (19)

$$\begin{aligned} \tau_0^{(1)}/\tau_0^{(2)} &= t_0^{(1)}/t_0^{(2)} \\ &= 1 + \frac{1}{2Ph} \left[ 1 + \frac{\ln \left\{ 1 + \frac{Bi}{1-\Delta(1+Bi)} \right\}}{\frac{Bi}{\Delta} + \frac{1}{\Delta^2} \ln \left\{ 1 + \frac{Bi}{1-1/\Delta} \right\}} \right]. \quad (20) \end{aligned}$$

As the second term in the bracket is always smaller than or equal to zero it follows that

$$t_0^{(1)}/t_0^{(2)} \leq 1 + 1/2Ph. \quad (21)$$

Equation (21) reflects the well-known fact that at large values of the phase-change parameter  $Ph$  the quasi-stationary theory will yield satisfactory results;  $Ph \geq 5$  is a commonly accepted criterion [24]. However, equation (20) also shows that for certain combinations of  $Bi$  and  $\Delta$  equation (19) may be equally satisfactory at smaller values of  $Ph$ . For instance, the same maximum possible error of 7.4% is found for both the following situations:

- (a)  $\Delta = 0.1, Bi = 2.0, Ph = 5.0$ ;
- (b)  $\Delta = 0.4, Bi = 0.5, Ph = 2.5$ .

Although such a trend is expected from purely physical reasoning it is now possible to actually quantify the range of the parameters where the quasi-stationary theory yields results of a specified minimum accuracy.

On the other hand it is clear that the true time of solidification will always lie in between those given by equations (18) and (19). It is therefore useful to have an estimate of the actual error involved in using either of the two results. The comparison with existing data and equations should also indicate the feasibility of a

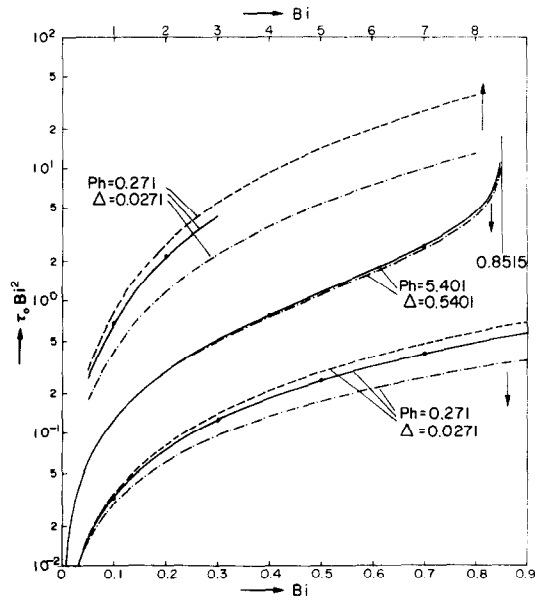


FIG. 3. Solidification time with finite heat transfer on both sides of the solid layer. Comparison between Stephan's analysis (—), the upper (---) and lower (-.-) bound and the weighted mean (●).

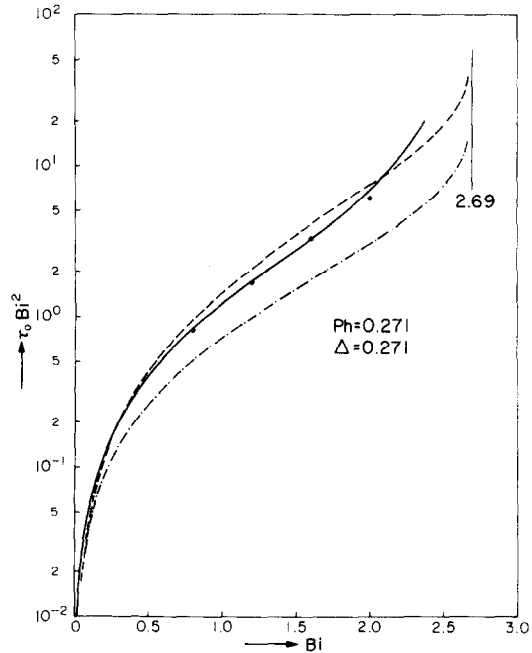


FIG. 4. Solidification time as in Fig. 3 but with large heat input from the melt.

weighted average of equations (18) and (19) for use in design and optimization studies.

For three different combinations of  $Ph$  and  $\Delta$  Stephan [15] presented results which are based on the assumption of a second-order polynomial for the temperature in the solid [13, 14]. His symbols  $1 - \xi^*$ ,  $\tau$ ,  $Ph$  and  $q^*$  are equivalent to  $Bi$ ,  $\tau_0 Bi^2$ ,  $Ph$  and  $\Delta/Ph$  respectively in the present notation. The comparison of the results is shown in Figs. 3 and 4 and a few comments may be justified.

A large phase-change parameter together with a sizeable heat flux from the melt results in a rather slow solidification process with a comparatively small final layer thickness ( $\tau_0 Bi^2$  and  $Bi$  are proportional to the absolute time  $\tau_0$  and layer thickness  $\delta_0$  respectively, see equation (12)). This favours the use of equation (18) rather than equation (19) although deviations are expected to remain small. Assuming that the parabolic approximation is correct at the parameter values present in Fig. 3 we find that for  $Ph = 5.401$  and  $\Delta = 0.5401$  the upper limit over-predicts by less than 0.4% and the lower limit under-predicts by less than 5%.

As expected these figures change substantially when  $Ph$  becomes small. The solid layer grows faster and to a much large thickness and the difference between upper and lower limit becomes increasingly large with longer times of solidification. Thus at  $Bi = 0.3$  ( $Ph = 0.271, \Delta = 0.0271$ ) the corresponding figures are +8.9% and -24.5% and they rise to +28.5% and -45.2% at  $Bi = 2.0$ . The latter parameter configuration would already be difficult or uneconomical to realize in practice but it is clear that in such extreme cases neither of the two limiting solutions is very satisfactory. Still, the simplicity and safety connected with equation (18) is attractive for engineering calculations. It is further concluded that smaller deviations will occur at values of  $Ph > 0.271$ .

Another extreme case is considered in Fig. 4 where the heat flux from the melt is large. It is seen that for a short as well as very long time Stephan's results exceed those from equation (18) by approximately 15 and 45% respectively. This is an unexpectedly large discrepancy and, until numerically exact results are available, no decision can be made as to which of the two concepts breaks down under such extreme conditions.

At intermediate values of the solidification time the comparison leads to much the same conclusions as with smaller heat flux from the melt.

3.2. Constant wall temperature ( $\Delta \neq 0, h_c^\circ \rightarrow \infty$ )

In this case we have to redefine the parameter  $\Delta$  because  $Bi \rightarrow \infty$  and  $\Delta \rightarrow 0$ . With the modified heat-flux parameter

$$\Delta^* = \Delta Bi$$

the upper and lower limit, for  $h_c^\circ \rightarrow \infty$ , become

$$\tau_0^{(1)} \Delta^{*2} = -(Ph + \frac{1}{2})(\Delta^* + \ln \{1 - \Delta^*\}) \quad (22)$$

and

$$\tau_0^{(2)} \Delta^{*2} = -Ph(\Delta^* + \ln \{1 - \Delta^*\}). \quad (23)$$

Here the maximum possible error is given by

$$\frac{t_0^{(1)}}{t_0^{(2)}} = 1 + \frac{1}{2Ph} \quad (21a)$$

and is seen to depend only on the phase-change parameter. In comparison with the previous case of "soft" cooling on the wall the solid layer will now grow faster which gives rise to a larger difference between the limits of solidification time.

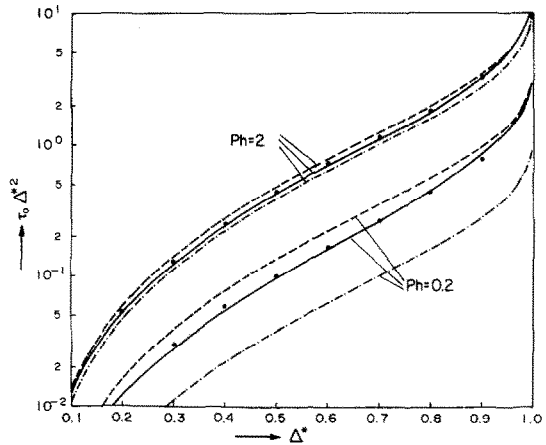


FIG. 5. Solidification time with constant wall temperature but convection from the melt. Comparison between numerical results (— [17]) the upper (---) and lower (-.-) bound and the weighted mean (●).

Beaubouef and Chapman [17] solved this problem numerically for three different values of the phase-change parameter, namely  $Ph = 0.2, 2.0$  and  $20.0$ . In Fig. 5 we have not included  $Ph = 20.0$  because from equation (21a) it is realized that the maximum possible error is only 2.5%. At  $Ph = 2.0$  the exact solution approaches the upper limit for short times (small layer thickness) and for very long times (slow solidification process); in between equation (22) over-predicts the true time by up to 10% whereas here the lower limit under-predicts by only 8%. Naturally, at  $Ph = 0.2$  the prediction errors become larger but one observes the same trends as in Figs. 3 and 4: For short or very long solidification processes equation (22) yields good agreement with the exact results, in between it predicts up to 40% larger solidification times although it still represents a much better approximation than equation (23), the quasi-stationary result.

A further comparison is possible with the approximate analysis of Lapadula and Mueller [25] which is based on the variational approach [10] and a parabolic temperature profile in the solid layer. Their result, in the present notation, is

$$\tau_0 \Delta^{*2} = f(Ph)[\Delta^* + \ln(1 - \Delta^*)] \quad (24)$$

with

$$-f(Ph) = \frac{2/Ph + 15Ph + 10}{5/Ph + 15}. \quad (25)$$

Equation (24) provides a generally better agreement with the numerical results of Beaubouef and Chapman (see the comparison in [17]) than does equation (22) but has the disadvantage of under-predicting the solidification time at small  $Ph$ -values and for short as well as very long processes.

On the other hand equation (25) can be compared directly with the corresponding expressions in equations (22) and (23). Figure 6 illustrates that with very good accuracy equation (25) can be replaced by

$$-f(Ph) = Ph + \frac{1}{2} \quad \text{for } Ph > 0.5 \quad (26)$$

This result already indicates that a weighted mean of upper and lower limit might lead to rather accurate results (see Section 4).

3.3 No heat flux from the melt

Again, the limiting solutions are easily derived from equations (18) and (19). With  $\Delta \rightarrow 0$  they become

$$\tau_0^{(1)} = Ph(1/Bi + \frac{1}{2}) + \frac{1}{4} + \frac{1}{2Bi} \left[ 1 - \frac{1}{Bi} \ln(1 + Bi) \right] \quad (27)$$

and

$$\tau_0^{(2)} = Ph(1/Bi + \frac{1}{2}). \quad (28)$$

The maximum possible error depends on both the phase-change parameter and the Biot number and varies between

$$t_0^{(1)}/t_0^{(2)} = 1 + \frac{1}{2Ph(1/Bi + \frac{1}{2})} \text{ for } Bi \rightarrow 0 \quad (29a)$$

and

$$t_0^{(1)}/t_0^{(2)} = 1 + \frac{1}{2Ph} \text{ for } Bi \rightarrow \infty. \quad (29b)$$

Since without heat input from the melt the solidification process is again faster than that of Section 3.1 the discrepancies between upper and lower bound are expected to be of the same order of magnitude as in Section 3.2. The comparison with numerical results [21] is seen in Fig. 7 and de-

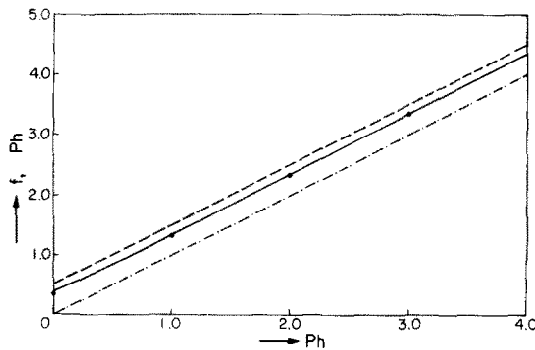


FIG. 6. Comparison between variational analysis (— [25]), upper (-----) and lower (-.-) bound and weighted mean (●).

monstrates clearly that neither equation (27) nor (28) supply useful approximations when  $Ph < 1$ . Prediction errors become increasingly larger with higher Biot numbers and in the limit of constant wall temperature ( $Bi \rightarrow \infty$ ) quite meaningless results are obtained. This last case apparently represents the fastest solidification process and is given by the Stefan-Neumann solution as shown graphically in Fig. 8. The upper and lower limiting solutions reduce to

$$\tau_0^{(1)} = Ph/2 + \frac{1}{4} \quad (30)$$

$$\tau_0^{(2)} = Ph/2 \quad (31)$$

and it is seen that at  $Ph = 0.25$  prediction errors of +50 and -50% occur.

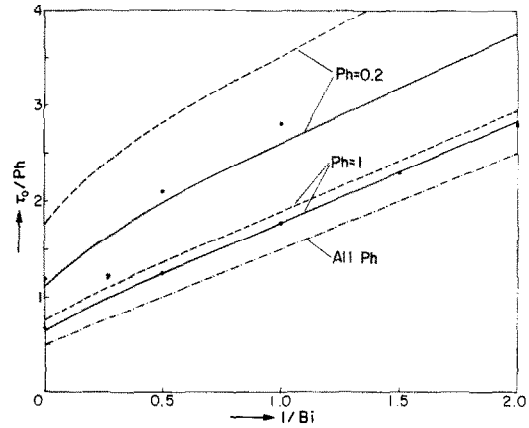


FIG. 7. Solidification time without heat flux from the melt. Comparison between numerical results (— [21]), upper (-----) and lower (-.-) bound and weighted mean (●).

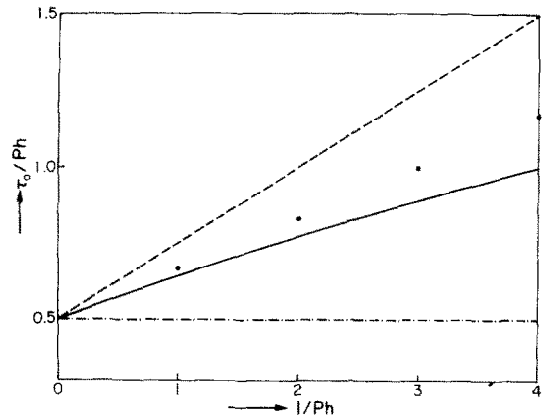


FIG. 8. Comparison between the Stefan-Neumann solution (—), upper (-----) and lower (-.-) bound and the weighted mean (●).

The comparison with nearly all the results published in the literature indicates that equation (18) and all its simplified versions provide a safe limit for the solidification time and over-predicts by 10% at most as long as the phase-change parameter is  $Ph \geq 1$ . However, for small  $Ph$ -values unacceptable errors occur and equation (18) on its own becomes of little use in engineering practice.

4. WORKING EQUATIONS

It is noticed that most of the exact results lie in the upper half of the envelope formed by the two limiting solutions, particularly when the phase-change parameter is small. This suggests that a linear combination of equations (18) and (19) might yield a simple working equation of sufficient accuracy over the entire range of practically important  $Ph$ -values. However, as with any

averaging procedure, it will be necessary to compromise between accuracy and simplicity of the result; we may also lose the advantage of predicting always safe results. The comparison of results as well as physical reasoning show that, with the same weighting factor applied throughout, prediction errors will occur in both directions and may, in places, be larger than with equation (18) alone. Still, just one simple formula to deal with all the discussed solidification processes has obvious advantages (see the Introduction) as long as a satisfactory accuracy is attained.

In Figs. 3–8 some arbitrary points are included which have been calculated according to the following averaging procedure:

$$\tau_0^{(3)} = \frac{2}{3}\tau_0^{(1)} + \frac{1}{3}\tau_0^{(2)}. \quad (32)$$

For the general problem equation (32) yields

$$\tau_0^{(3)}Bi^2 = \left(-Ph - \frac{1}{3}\right) \left( \frac{Bi}{\Delta} + \frac{1}{\Delta^2} \ln \left\{ 1 + \frac{Bi}{1 - 1/\Delta} \right\} \right) - \frac{1}{3} \ln \left\{ 1 + \frac{Bi}{1 - \Delta(1 + Bi)} \right\} \quad (33)$$

and, excluding the sections of Fig. 4 which have been discussed before, we find excellent agreement with Stephan's results; a maximum error of approximately  $-2.5\%$  occurs at the very small Biot numbers.

With the same weighting factors one obtains for the solidification time at constant wall temperature

$$\tau_0^{(3)}\Delta^{*2} = -(Ph + \frac{1}{3})[\Delta^* + \ln(1 - \Delta^*)] \quad (34)$$

which is identical to equation (26) and in good agreement with the analytical result of Lapadula and Mueller. The comparison of equation (34) with numerical results (see Fig. 5) indicates errors of up to  $5\%$  as long as the solid layer does not grow to more than  $85\%$  of its final thickness at infinite time.

Without heat flux from the melt and equation (32) we get

$$\tau_0^{(3)} = Ph \left( \frac{1}{Bi} + \frac{1}{2} \right) + \frac{1}{6} + \frac{1}{3Bi} \left( 1 - \frac{1}{Bi} \ln \{ 1 + Bi \} \right) \quad (35)$$

and

$$\tau_0^{(3)} = Ph/2 + \frac{1}{6} \quad (\text{for } Bi \rightarrow \infty). \quad (36)$$

Again, the results are shown only at a few points (Figs. 7 and 8). It is seen that in both these cases of relatively fast growth of the solid layer an equal weighting of the limiting solution would result in an improved accuracy. In fact, the Stefan–Neumann solution is very accurately represented by

$$\tau_0 = Ph/2 + \frac{1}{8} \quad (\leq -3.5\%).$$

But for the sake of generality we prefer to leave the weighting factors as specified by equation (32). This leads to maximum errors of approximately  $+9\%$  (Fig. 7) and  $+15\%$  (Fig. 8) at  $Ph = 0.25$ ; smaller errors arise with  $Ph > 0.25$ .

## 5. CONCLUSIONS

The lower limit for the solidification time of a slab-shaped body is given by the well established quasi-stationary solution. Based on an energy balance an analytic solution has been obtained which seems to constitute an upper limit for the solidification time.

The solution is valid for the most commonly encountered boundary conditions, i.e. finite heat transfer from the melt to the solid layer and finite heat transfer or constant wall temperature on the cooling wall. In comparison with literature results the prediction error is less than  $10\%$  for  $Ph \geq 1$ . Smaller values of the phase-change parameter lead to a large overestimation but in such cases a weighted average of both limiting solutions proves to be useful. With a  $\frac{2}{3}$  and  $\frac{1}{3}$  weighting of the upper and lower limit respectively a particularly simple result is obtained which represents the published data with a relatively high accuracy. Over a wide range of parameter combinations the errors are less than  $5\%$  and only for extremely long or extremely fast solidification processes will the prediction error rise to  $10$  or  $15\%$ . In view of other short-comings of the analysis, such as the neglected temperature dependency of the physical properties, the present results are believed to be sufficiently accurate.

It is easily verified that the same general principle applies also to melting problems as long as the boundary conditions are equivalent to the ones discussed here. Thus, a semi-infinite solid at fusion temperature which is heated from a source of constant temperature and through a stagnant layer of liquefied material [26] can be treated by the equations of Section 3.3; on the other hand, the treatment of the classical ablation problem would require additional assumptions concerning the penetration thickness.

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#### NOTE ADDED IN PROOF

Equation (18) is, in fact, an upper bound for the solidification time. The proof will appear shortly.

#### SOLUTION SIMPLE ET VRAISEMBLABLEMENT DE SECURITE POUR LE PROBLEME DE STEFAN GENERALISE

**Résumé**—La solidification ou la fusion d'un corps en forme de plaque peut se produire sous des conditions diverses à chaque frontière du solide. Seule la situation la plus simple a été soumise à une analyse stricte tandis que le problème plus général a été résolu à l'aide de théories approchées et par simulation numérique. Cependant, la plupart des résultats ne peuvent convenir aux buts industriels, tels que l'optimisation des dispositifs de changement de phase, les théories prévoyant des taux de changement de phase qui sont trop faibles dans certains cas et trop forts dans d'autres.

Le but du présent article est de fournir un résultat sous forme analytique simple qui conduise à une prévision des temps de congélation surestimés (sécurité) pour toute combinaison des paramètres d'intérêt pratique. On montre ensuite que ce résultat peut être utilisé pour formuler des équations très précises pour l'exploitation.

#### EINE EINFACHE UND AUF DER SICHEREN SEITE LIEGENDE LÖSUNG DES VERALLGEMEINERTEN STEFAN-PROBLEMS

**Zusammenfassung**—Das Erstarren oder Schmelzen eines plattenförmigen Körpers kann unter verschiedenen Bedingungen auf beiden Seiten des Festkörpers erfolgen. Lediglich der einfachste Fall konnte bislang streng analytisch gelöst werden, während das verallgemeinerte Problem näherungsweise bzw. numerisch gelöst wurde. Jedoch sind die meisten Ergebnisse ungeeignet für Ingenieurszwecke, z.B. für die Optimierung von Anlagen, in denen ein Phasenwechsel erfolgt. Die Theorien ergeben Phasenwechselgeschwindigkeiten, die in einigen Fällen zu klein, in einigen zu groß sind. Das Ziel dieser Arbeit ist es, ein Ergebnis zu bieten, das eine einfache analytische Struktur aufweist, und das für alle in der Praxis auftretenden Parameter zu große (sichere) Gefrierzeiten liefert. Desweiteren wird gezeigt, daß dieses Ergebnis dazu verwendet werden kann, Berechnungsgleichungen hoher Genauigkeit aufzustellen.

#### ПРОСТОЕ И ДОСТАТОЧНО НАДЕЖНОЕ РЕШЕНИЕ ОБОБЩЕННОЙ ЗАДАЧИ СТЕФАНА

**Аннотация** — Затвердевание или плавление тела, имеющего форму пластины, может происходить при различных условиях на обеих границах твердого тела. Строгий анализ проводился только для простейшего случая, в то время как более общая задача решалась с помощью приближенных методов и путем численного моделирования. Однако, большинство из полученных результатов не подходит для инженерных целей, как например, оптимизации параметров оборудования при фазовых превращениях в нем, а теоретический расчет дает скорости фазовых изменений, которые или слишком малы в одних случаях, или слишком велики в других. Цель данной работы состоит в получении простой аналитической зависимости, которая позволит рассчитать время для достаточного длительного процесса замораживания при всех комбинациях параметров, представляющих практический интерес. Показано, что этот результат может быть использован для получения рабочих уравнений большой точности.